1. Show that the following Hoare triple holds [5 pts]

\{ x \geq 2 \} \quad x := x - y + 3 \quad \{ x + y \geq 0 \}

2. For each of the statements below, indicate whether the statement is true or false. If true, give a proof. If false, give a counterexample.

**Constants:**
(a) \{ \text{false} \} \text{ next } \{ Q \} [2 pts]
(b) \{ P \} \text{ next } \{ \text{true} \} [2 pts]
(c) \text{true} \text{ next } \text{false} [2 pts]

**Junctivity:**
(a) \( (P_1 \text{ next } Q_1) \land (P_2 \text{ next } Q_2) \implies (P_1 \land P_2) \text{ next } (Q_1 \land Q_2) \) [3 pts]
(b) \( (P_1 \text{ next } Q_1) \land (P_2 \text{ next } Q_2) \implies (P_1 \lor P_2) \text{ next } (Q_1 \lor Q_2) \) [3 pts]

**Weakening:**
(a) \( P \text{ next } Q) \land [Q \implies Q'] \implies (P \text{ next } Q') [3 pts]

3. For each of the statements below, indicate whether the statement is true or false. If true, give a proof. If false, give a counterexample.

(a) \( \text{stable}(P) \land \text{stable}(Q) \implies \text{stable}(P \land Q) \) [5 pts]
(b) \( \text{stable}(P) \land \text{stable}(Q) \implies \text{stable}(P \lor Q) \) [5 pts]
(c) \( \text{stable}(P) \land [P \implies P'] \implies \text{stable}(P') \) [5 pts]

4. Given \( N \) agents indexed \( 0, \ldots, N-1 \), where \( N > 2 \). Each agent \( j \) has a real number \( x_j \). Let \( A \) be the average of the \( x_j \) values and let \( V \) be the variance.

**Initial Values** The initial values of \( x_j \) are arbitrary (but finite). Let \( x_j^{(0)} \) be the initial value of \( x_j \), and let \( A^{(0)} \) be the average of the initial values of \( x_j \).
**Iteration**  Pick any two agents $i$ and $j$ non-deterministically, with weak fairness. Set:

$$x_i, x_j := (x_i + x_j)/2, (x_i + x_j)/2$$

(a) Prove that the variance never increases. More formally, show that:

$$\forall K : \text{stable}(V \leq K)$$

(b) Prove the following Hoare triple.

$$\{x_i \neq x_j \land V = K\} \ x_i, x_j := (x_i + x_j)/2, (x_i + x_j)/2\} \ {V < K}$$

This implies that performing the the assignment action corresponding to any $x_j, x_k$ that are not equal decreases the variance $V$.

---

5. [Nondeterministic Iteration – Shortest Paths] Somebody proposes the following algorithm to find the shortest path between every pair of vertices in a finite directed graph. Let $W$ be the edge-weight matrix, i.e., $W[j, k]$ is the weight of edge $(j, k)$. Weights are real numbers. Assume that the graph is completely connected, and therefore $W[j, k]$ exists for all $j, k$. Also $W[j, j] = 0$ for all $j$. The graph has no cycles of negative weight.

Let $D$ be a matrix with the same dimensions as $W$. The proposed algorithm is as follows:

(a) **initially:** $D = W$

(b) There is a command for every triple $(i, j, k)$ of vertices, and the command is:

$$\text{IF } D[i, k] > D[i, j] + D[j, k] \text{ THEN } D[i, k] := D[i, j] + D[j, k] \quad (1)$$

We will complete a part of the proof for the correctness of the algorithm. The claim is that the algorithm will terminate with $D$ being the matrix of shortest path lengths, i.e., $D[j, k]$ will be the length of the shortest path from vertex $j$ to vertex $k$.

Prove that the conjunction of following two predicates is an invariant for the above program.

(a) $\forall j, k : D[j, k] \leq W[j, k]$

and

- for all $j, k$, $D[j, k]$ is the length of some path from vertex $j$ to vertex $k$

(b) Does the program (algorithm) have a fixed point at which the invariant property from the previous part holds. If yes, propose the fixed point and prove it. Recall that a state is a fixed point if there exists no action in the program that changes the state.